Performance metrics for multiobjective optimization evolutionary algorithms

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Abstract. In the recent past many algorithms for multiobjective optimization have been proposed. To evaluate performances of these algorithms some measures of performances are needed. Many metrics of algorithms performances have been proposed. The existing performance metrics are briefly reviewed. Two metrics computing the convergence towards the Pareto front and the solution diversity on the Pareto front are proposed.

Keywords: evolutionary algorithms, multiobjective optimization, Pareto front, performance metrics

1. Introduction

In the last years many multiobjective optimizations algorithms (MOA) have been proposed. To compute the performances of these algorithms some measures of performance were also introduced. Most of them are applied to the final nondominated set. It is now established that more than one metrics are necessary to evaluate the performances of the multiobjective evolutionary algorithms. Zitzler [5] has recently shown that for an M-objective optimization problem, at least M performance metrics must be used.

According to Deb [1] the existing performance metrics can be classified into three classes: metrics for convergence, metrics for diversity and metrics for both convergence and diversity.

Some of more recent and important metrics of performance are reviewed in the next section. Two new metrics one for convergence and one for diversity are introduced in Section 3.

2. Performance metrics: a review

Here we classify the here measures for evolutionary algorithms performances in two major classes:

- convergence metrics – evaluate how far from the true Pareto front solutions obtained in final population are;
- diversity metrics – evaluate scatter of solutions in the final population on the Pareto front.

Each of them is described in detail bellow.

2.1. **Metrics for convergence.** Many metrics for measuring the convergence of a set of nondominated solutions towards the Pareto front have been proposed. Almost all of these metrics were constructed in order to directly compare two sets of nondominated solutions. There are also approaches which compare a set of nondominated solutions with a set of Pareto optimal solutions if the true Pareto front is known.

In what follows we review some existing metrics for convergence.

2.1.1. *Metric S.* The S metric has been introduced by Zitzler in [4] and improved in [5]. The S metric measures how much of the objective space is dominated by a given nondominated set A.

Definition 1 (Size of the dominated space). *Let X be set of decision vectors for the considered problem and let $A = \{x_1, x_2, \ldots, x_t\} \subseteq X$ be a set of t decision vectors. The function $S(A)$ gives the volume enclosed by the union of the polytopes $p_1, p_2, \ldots, p_t$, where each p_i is formed by the intersection of the following hyperplanes arising out of x_i, along with the axes: for each axis in the objective space there exist a hyperplane perpendicular to the axis and passing through the point $(f_1(x_i), f_2(x_i), \ldots, f_k(x_i))$.*

Example 2. *In the two-dimensional case, each p_i represents a rectangle defined by the points (0,0) and $(f_1(x_i), f_2(x_i))$. An example for the two-dimensional case is presented in fig. 1.*

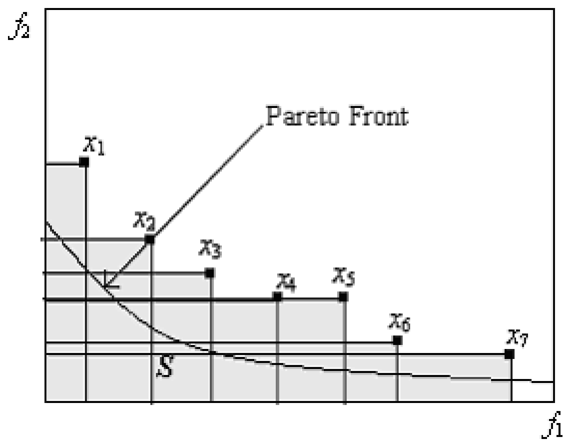

FIGURE 1. The metric$S$ for the case of two objective functions and 7 decision vectors $(x_1, x_2, \ldots, x_7)$ for a minimization problem.

2.1.2. Metric C The metric C, like the metric S, was introduced by Zitzler in [4] and improved in [5]. Using the metric C two sets of nondominated solutions can be compared to each other.

Definition 3 (Coverage of two sets). *Let X be the set of decision vectors for the considered problem and let $A, B \subseteq X$ be two sets of decision vectors. The function C maps the ordered pair (A, B) into the interval [0,1]:*

$$C(A, B) = \frac{|\{b \in B \ / \ \exists a \in A{:}a \succeq b\}|}{|B|}.$$

Remark 4. *(i) The value $C(A, B) = 1$ means that all decision vectors in B are dominated by A.*
(ii) The value $C(A, B) = 0$ represent the situation when none of the points in B are dominated by A.
(iii) $C(A, B)$ is not necessary equal to $1 - C(B, A)$.

Example 5. *There are situations when the metric C cannot decide if an obtained front is better than the other. Let us suppose that front 1 correspond to a set A and front 2 to a set B.*

In fig. 2, the surface covered by the front 1 is equal to the surface covered by the front 2 but front 2 is closer to the Pareto optimal front than front 1. In this situation (and in other situations similar with this) the C metric is not applicable. To eliminate this shortcoming a new metric – D metric – was proposed.

Definition 6 (Coverage difference of two sets). *Let $A, B \subseteq X$ be two sets of decision vectors. The size of the space dominated by A and not dominated by B (regarding the objective space) is denoted by $D(A, B)$ and is defined as:*

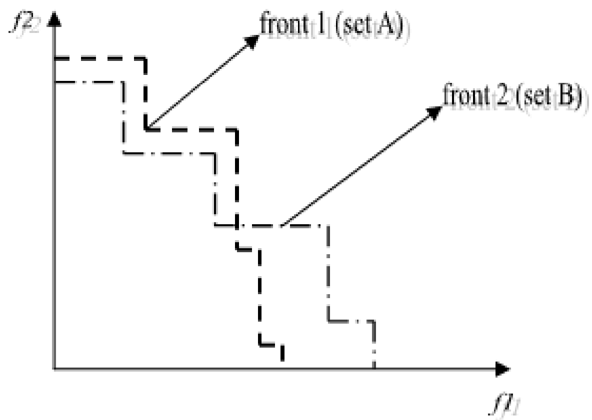

FIGURE 2. An example when the metric C can not decide between front 1 and front 2 (the surface covered by the front 1 is equal to the surface covered by the front 2).

$$D(A, B) = S(A + B) - S(B),$$

where $S(A)$ is defined above.

Example 7. *The metric D can be used to solve the inconvenience of Example 2. Consider the notation in fig. 3.*

By applying the metric D the followings equalities are obtained

$$S(A + B) = \alpha + \beta + \gamma;$$

$$S(A) = \alpha + \gamma;$$

$$S(B) = \alpha + \beta.$$

The metric D for this example is expressed below

$$D(A, B) = \gamma;$$

$$D(B, A) = \beta.$$

From

$$D(A, B) < D(B, A)$$

it follows that the front 2 dominates the front 1.

2.2. **Diversity metrics.** In this section the most frequently used metric for diversity is described.
2.2.1 A diversity metric In this section we consider a metric for diversity proposed by Deb in [2]. The obtained nondominated points at each generation are projected on a suitable hiperplan. The plan is divided into a number of small grids (($M - 1$) dimensional boxes, M being the number of objectives). The diversity metric is defined according to on whether each grid contains an obtained nondominated point or not. The best possible result is obtained if all grids are represented with at least one point. If some grids are not represented by a nondominated point the diversity is poor.

Remark 8. *For greater number of objectives the value function will be difficult to define.*

3. Two new metrics for convergence and diversity

In this section two metrics - one for evaluate the convergence to the Pareto set and the other to determinate the spread of the solutions on the Pareto set are proposed.

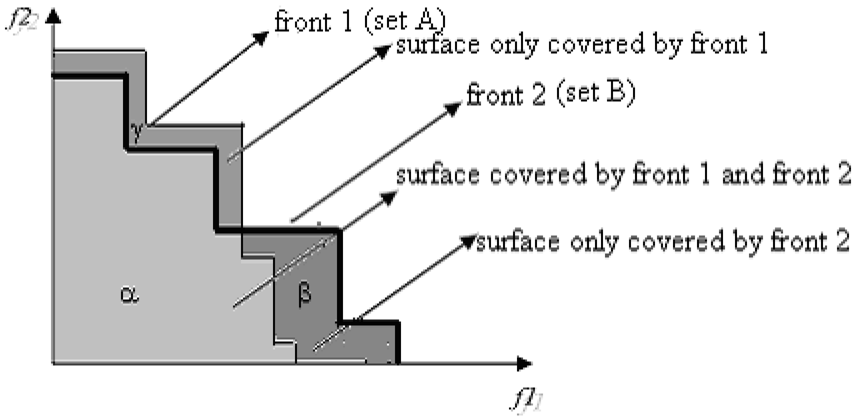

FIGURE 3. Example of difference between C metric and D metric for the considered fronts front 1 and front 2.

3.1. **New convergence metric.** Assume that the Pareto front is known. Let us denote by P a set of Pareto optimal solutions.

For each individual from the final population distance (Euclidian distance or other suitable distance) to all points of $P$is computed. The minimum distance is kept for each individual. The average of these distances represents the measure of convergence to the Pareto front.

3.2. **New diversity metric.** For each individual from the final population we consider the point from the set of Pareto optimal points P situated at the minimal distance. Several concepts of distance to a set may be considered. Here we consider $d(x,\, P)$ as being

$$d(x,\, P) = \min_{y \in P} d(x, y).$$

We called each such point from Pa *marked* point. The total number of different marked points from P over the size of P represents the diversity metric.

Remark 9. *These two metrics have a low computational cost. These metrics can be applied to high dimensional spaces.*

4. Conclusions

Many metrics have been proposed in the last years. Most of them calculate the convergence to an obtained set of solutions to the true Pareto front. The others measure the diversity of the obtained set of solutions on the Pareto front. We can not say that one metric is the best. Some of these metrics are preferred considering some aspects; the others, for the other aspects. Some of them are preferred to the others by considering the computation complexity. For different classes of problems different types of metrics can be preferred.

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